**RFT 12.1.1 – Explicit Numerical Verification of Anomaly Cancellation and Fermion Representation Stability**

This supplement provides a detailed check of gauge anomaly cancellation and fermion content consistency for one Standard Model (SM) generation, in support of RFT 12.1. We explicitly verify that all chiral gauge anomalies cancel out numerically and that the fermion representation content avoids any global $SU(2)$ (Witten) anomaly. A summary of the fermion representations and hypercharges used is provided at the end for reference.

**1. Numerical Anomaly Cancellation**

A single generation of SM fermions includes one $SU(2)\_L$ doublet of quarks ($Q\_L$), one $SU(2)\_L$ doublet of leptons ($L\_L$), and three $SU(2)\_L$ singlet fermions ($u\_R$, $d\_R$, $e\_R$). Their gauge quantum numbers (color $SU(3)\_C$ representation, weak isospin $SU(2)\_L$ representation, and hypercharge $U(1)\_Y$) are the standard ones listed in Section 3. We check each possible chiral gauge anomaly involving $U(1)\_Y$ and the non-Abelian gauge factors, as well as the mixed gravitational anomaly:

* **$[SU(2)\_L]^2 U(1)\_Y$ anomaly:** Requires the sum of hypercharge $Y$ over all left-handed fermions in $SU(2)\_L$ doublets to vanish​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203).
* **$[SU(3)\_C]^2 U(1)\_Y$ anomaly:** Requires the sum of $Y$ over all left-handed fermions carrying color charge (triplets of $SU(3)\_C$) to vanish​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203).
* **$[U(1)\_Y]^3$ anomaly:** Requires the sum of $Y^3$ over all left-handed fermions to vanish​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=X%20f%20Y%20,2%20%12%E2%88%92%201%208%20%13).
* **$U(1)\_Y$–gravitational anomaly:** Requires the sum of $Y$ over all left-handed fermions (weighted by multiplicities) to vanish​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203). (This is the condition that the $U(1)\_Y$ hypercharge current has zero trace, ensuring no mixed gauge-gravity anomaly.)

In the SM, these conditions are satisfied for each generation​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203)​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=X%20f%20Y%20,2%20%12%E2%88%92%201%208%20%13). We demonstrate this by summing the contributions of each chiral fermion. In the calculations below, we account for **multiplicity** factors due to color and weak isospin degrees of freedom (since each color state or isospin component is a distinct left-chiral fermion in the anomaly sum). We use the conventional hypercharge assignments where electric charge $Q = T\_3 + \tfrac{1}{2}Y$ (so, e.g., $Q\_L$ has $Y=+1/3$, $L\_L$ has $Y=-1$, etc.).

**1.1 $[SU(2)\_L]^2 U(1)\_Y$ Anomaly**

Only $SU(2)\_L$ **doublets** contribute to the $[SU(2)]^2 U(1)$ anomaly. In one generation, these are:

* **Quark doublet $Q\_L(3,2)\_{Y=+1/3}$:** 3 colors × 1 doublet, hypercharge $+\tfrac{1}{3}$ per doublet.
* **Lepton doublet $L\_L(1,2)\_{Y=-1}$:** 1 color × 1 doublet, hypercharge $-1$ per doublet.

Right-handed fermions are $SU(2)\_L$ singlets and do not enter this anomaly. Summing the hypercharge of each $SU(2)$ doublet (including color copies):

* $Q\_L$ contributes $3 \times \left(+\frac{1}{3}\right) = +1$.
* $L\_L$ contributes $1 \times \left(-1\right) = -1$.

**Sum:** $+1 + (-1) = 0$.

This vanishing sum confirms that the $[SU(2)\_L]^2U(1)\_Y$ anomaly cancels. In other words, the total hypercharge of all left-handed $SU(2)$ doublet fermions is zero, as required.

**1.2 $[SU(3)\_C]^2 U(1)\_Y$ Anomaly**

Only fermions carrying color charge ($SU(3)\_C$ non-singlets) contribute to the $[SU(3)]^2 U(1)$ anomaly. In one generation, these are the quarks: the left-handed quark doublet $Q\_L$ (which has two $SU(2)$ components) and the two right-handed quarks $u\_R$ and $d\_R$. We must ensure that the net hypercharge-weighted color charge of left-handed vs. right-handed quarks cancels out. Summing hypercharge contributions for all color triplet fermions:

* **Left-handed color triplets:** The quark doublet $Q\_L(3,2)\_{Y=+1/3}$ contains *two* weak isospin components (up and down), each a color triplet with $Y=+1/3$. Together they contribute $2 \times \left(+\frac{1}{3}\right) = +\frac{2}{3}$.
* **Right-handed color triplets:** $u\_R(3,1)*{Y=+4/3}$ contributes $+\frac{4}{3}$, and $d\_R(3,1)*{Y=-2/3}$ contributes $-\frac{2}{3}$. The sum for right-handed quarks is $+\frac{4}{3} + \left(-\frac{2}{3}\right) = +\frac{2}{3}$.

We find that the total hypercharge sum for left-handed color triplets is $+2/3$, equal to the sum for right-handed color triplets ($+2/3$). The **difference** between left- and right-handed contributions is zero, indicating the $[SU(3)\_C]^2 U(1)\_Y$ anomaly cancels. Equivalently, one can say the hypercharge contributions of the $Q\_L$ doublet balance those of $u\_R$ and $d\_R$ in the triangle diagram​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203).

**1.3 $[U(1)\_Y]^3$ Anomaly**

The cubic hypercharge anomaly involves the sum $\sum\_f Y\_f^3$ over all left-handed Weyl fermions $f$. We compute the contribution of each fermion state (counting each distinct color and isospin state separately):

* **Quark doublet $Q\_L(3,2)\_{Y=+1/3}$:** There are 6 left-chiral states here (3 colors × 2 weak components), each with $Y=+1/3$. Total contribution: $6 \times \left(\frac{1}{3}\right)^3 = 6 \times \frac{1}{27} = \frac{6}{27} = \frac{2}{9}$.
* **Lepton doublet $L\_L(1,2)\_{Y=-1}$:** 2 states (e.g. $\nu\_L$ and $e\_L$), each with $Y=-1$. Contribution: $2 \times (-1)^3 = 2 \times (-1) = -2$.
* **Right-handed up quark $u\_R(3,1)\_{Y=+4/3}$:** 3 states (color triplet), each with $Y=+4/3$. Contribution: $3 \times \left(\frac{4}{3}\right)^3 = 3 \times \frac{64}{27} = \frac{192}{27} = \frac{64}{9}$.
* **Right-handed down quark $d\_R(3,1)\_{Y=-2/3}$:** 3 states, each with $Y=-2/3$. Contribution: $3 \times \left(-\frac{2}{3}\right)^3 = 3 \times \left(-\frac{8}{27}\right) = -\frac{24}{27} = -\frac{8}{9}$.
* **Right-handed charged lepton $e\_R(1,1)\_{Y=-2}$:** 1 state with $Y=-2$. Contribution: $1 \times (-2)^3 = -8$.

Now we sum all these contributions. It is convenient to sum left-handed and right-handed parts separately:

* **Left-hand side (sum of $Y^3$ for $Q\_L$ and $L\_L$):** $\displaystyle \frac{2}{9} + (-2) = -\frac{16}{9}$.
* **Right-hand side (sum of $Y^3$ for $u\_R$, $d\_R$, $e\_R$):** $\displaystyle \frac{64}{9} + \left(-\frac{8}{9}\right) + (-8) = \frac{64}{9} - \frac{8}{9} - \frac{72}{9} = -\frac{16}{9}.$

We find the left-hand and right-hand sums are **equal** ($-16/9$ each). Therefore, the total $Y^3$ sum (left + right, or equivalently the difference left $-$ right) is zero: $-\frac{16}{9} - \left(-\frac{16}{9}\right) = 0$. This confirms that the $[U(1)\_Y]^3$ anomaly cancels out **exactly** for one generation​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=X%20f%20Y%20,2%20%12%E2%88%92%201%208%20%13). The equality of the two sides reflects the well-known cancellation between quark and lepton hypercharge cubes in the SM​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=X%20f%20Y%20,2%20%12%E2%88%92%201%208%20%13).

*(For a single generation, one can also verify the sum directly: $\frac{2}{9} - 2 + \frac{64}{9} - \frac{8}{9} - 8 = \frac{2 + 64 - 8}{9} - 10 = \frac{58}{9} - \frac{90}{9} = -\frac{32}{9}$, and note that in a gauge theory context the contributions of right-handed fermions enter with opposite sign to those of left-handed fermions. Hence the anomaly measure is proportional to $-\frac{32}{9}$ for left vs. right, which vanishes when properly accounting for chirality. The method above of comparing left and right sums is the clearest way to see the cancellation.)*

**1.4 $U(1)\_Y$–Gravitational Anomaly**

The $U(1)\_Y$–gravity mixed anomaly involves a triangle diagram with two external gravitational legs and one $U(1)\_Y$ gauge leg. Cancellation requires the net hypercharge (first moment) of all chiral fermions to be zero​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203). This is equivalent to $\sum\_f Y\_f = 0$ when summing over all left-handed fermion states. Using the multiplicities and hypercharges from above:

* **Quark doublet $Q\_L$:** $6$ states with $Y=+1/3$ $\to$ total $Y = 6 \times \frac{1}{3} = +2$.
* **Lepton doublet $L\_L$:** $2$ states with $Y=-1$ $\to$ total $Y = 2 \times (-1) = -2$.
* **Right-handed $u\_R$:** $3$ states with $Y=+4/3$ $\to$ total $Y = 3 \times \frac{4}{3} = +4$.
* **Right-handed $d\_R$:** $3$ states with $Y=-2/3$ $\to$ total $Y = 3 \times \left(-\frac{2}{3}\right) = -2$.
* **Right-handed $e\_R$:** $1$ state with $Y=-2$ $\to$ total $Y = -2$.

Summing everything: $+2 + (-2) + 4 + (-2) + (-2) = 0$. We can again group left vs. right: left-hand total $Y = +2 + (-2) = 0$, right-hand total $Y = 4 + (-2) + (-2) = 0$. In either case, the **net sum is zero**, satisfying the $U(1)\_Y$–gravity anomaly cancellation condition​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203). This means the **trace of the hypercharge** over all chiral fermions vanishes, a requirement for the theory to be free of any gravitational non-conservation of $U(1)\_Y$ current.

All the above results confirm that a single generation of SM fermions is free of gauge and gravitational anomalies. These numerical checks illustrate the delicate balance between quark and lepton contributions in the SM. In fact, the hypercharge assignments of the SM are essentially fixed (up to overall normalization) by the requirement of anomaly cancellation​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%282,in%20order%20not%20to%20introduce). Any new chiral fermions introduced beyond this content would have to be assigned quantum numbers carefully to preserve these cancellations.

**2. Witten Anomaly Check (Global $SU(2)$ Anomaly)**

In addition to the perturbative anomalies above, we must ensure the absence of the **Witten anomaly** for $SU(2)\_L$. A global $SU(2)$ anomaly occurs if the number of left-handed fermion doublets is **odd**, since $\pi\_4(SU(2))=\mathbb{Z}\_2$ implies an obstruction for an odd number of spin-$\frac{1}{2}$ doublets. In the Standard Model, each generation contains the following $SU(2)\_L$ doublets:

* 3 quark doublets (from $Q\_L$, one for each color $r,g,b$)
* 1 lepton doublet ($L\_L$)

This makes a total of **4** $SU(2)\_L$ doublets per generation, which is an even number. Having an even number of $SU(2)$ doublets ensures there is no global $SU(2)$ anomaly​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Witten%20anomaly%3A%20even%20number,f%29%20%3D%203). In other words, the Witten anomaly is avoided. We can see that the three colored quark doublets alone (which would be 3 doublets, an odd number) would *not* be anomaly-free – it is the inclusion of the lepton doublet that provides the fourth to make the total even​[damtp.cam.ac.uk](https://www.damtp.cam.ac.uk/user/tong/sm/standardmodel5.pdf#:~:text=doublets,we%20must%20also%20have%20LL). This interdependence between quarks and leptons is a striking consistency check: the theory would be inconsistent if we tried to omit the lepton doublet or if hypercharges were assigned differently, underscoring the “fermion representation stability” of the SM content.

**Conclusion:** One generation of Standard Model fermions passes all anomaly cancellation tests. There are no gauge anomalies ($[SU(2)]^2U(1)$, $[SU(3)]^2U(1)$, $[U(1)]^3$ all cancel) and no mixed gravitational anomaly (sum of $Y$ vanishes)​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203)​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=X%20f%20Y%20,2%20%12%E2%88%92%201%208%20%13). The even number of $SU(2)$ doublets ensures no Witten anomaly​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Witten%20anomaly%3A%20even%20number,f%29%20%3D%203). These calculations provide numerical transparency to the claim that the **baseline fermion content** of RFT 12.1 (which mirrors the SM content) is anomaly-free and internally consistent.

**3. Fermion Content Review (One Generation, SM Basis)**

For completeness, we summarize the chiral fermion content and quantum numbers assumed in this version of the theory. **Table 1** lists the representation of each fermion field in one SM generation under the gauge group $SU(3)\_C \times SU(2)\_L \times U(1)\_Y$. Only the Standard Model fermions are included (we do **not** introduce any new exotic fermions from the scalaron–twistor sector in this baseline model). Each entry represents a left-handed Weyl spinor field; right-handed fields are listed separately as they transform as conjugate representations under $SU(2)\_L$. The hypercharge values are chosen to satisfy the anomaly cancellation conditions verified above​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%282,in%20order%20not%20to%20introduce).

**Table 1 – Standard Model Fermion Content (1 Generation)**

| **Fermion** | **Representation $(SU(3)\_C,,SU(2)*L)*{Y}$** | **Description** |
| --- | --- | --- |
| $Q\_L = (u\_L,\ d\_L)$ | $(3,\ 2)\_{+1/3}$ | Left-handed quark doublet (up-type and down-type quarks). 3 colors, weak isospin doublet. |
| $u\_R$ | $(3,\ 1)\_{+4/3}$ | Right-handed up-type quark (singlet under $SU(2)\_L$). 3 colors. |
| $d\_R$ | $(3,\ 1)\_{-2/3}$ | Right-handed down-type quark (singlet under $SU(2)\_L$). 3 colors. |
| $L\_L = (\nu\_L,\ e\_L)$ | $(1,\ 2)\_{-1}$ | Left-handed lepton doublet (neutrino and charged lepton). Weak isospin doublet, no color. |
| $e\_R$ | $(1,\ 1)\_{-2}$ | Right-handed charged lepton (electron or muon/tau equivalent). Singlet under $SU(2)\_L$. |

*Notes:* The hypercharge $Y$ is given in the convention $Q = T\_3 + \frac{1}{2}Y$. All fields are left-handed two-component fermions. (Right-handed fields like $u\_R$, $d\_R$, $e\_R$ are CP-conjugated to left-handed Weyl spinors for the purpose of representation assignment.) No right-handed neutrino ($\nu\_R$) is included in the minimal content, which is consistent with the Standard Model – $\nu\_L$ has no partner and remains massless in the absence of additional mechanisms. Each of the above fields appears once per generation; the Standard Model has three generations that replicate this content. As demonstrated in Section 1, this set of representations satisfies all anomaly cancellation requirements for each generation​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Gauge%20%C3%97%20gravity%20anomalies,f%29%20%3D%203)​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=X%20f%20Y%20,2%20%12%E2%88%92%201%208%20%13), and as discussed in Section 2, the total count of four $SU(2)$ doublets per generation guarantees no Witten anomaly​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%E2%80%A2%20Witten%20anomaly%3A%20even%20number,f%29%20%3D%203).

Overall, the fermion content is **stable** in the sense that it does not require any additional fermions to cancel anomalies – the given set is self-consistent. This justifies using only the SM fermions in RFT 12.1’s construction. Any extension of the fermion sector (e.g. introducing new chiral fields from a scalaron–twistor construction) would need a similarly thorough anomaly analysis to ensure the theory’s consistency​[itp.kit.edu](https://www.itp.kit.edu/_media/courses/ss2018/susy/susy.pdf#:~:text=%282,in%20order%20not%20to%20introduce). Here, by explicitly verifying each anomaly cancellation by numerical summation, we provide a firm foundation that the baseline RFT model’s fermion sector (identical to the SM’s) is anomaly-free and thus a mathematically consistent starting point for further theoretical developments.